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D-instanton Sums for Matter Hypermultiplets

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Abstract

We calculate some non-perturbative (D-instanton) quantum corrections to the moduli space metric of several ($n > 1$) identical matter hypermultiplets for the type-IIA superstrings compactified on a Calabi-Yau threefold, near conifold singularities. We find a non-trivial deformation of the (real) $4n$-dimensional hypermultiplet moduli space metric due to the infinite number of D-instantons, under the assumption of n tri-holomorphic commuting isometries of the metric, in the hyper-Kähler limit (i.e. in the absence of gravitational corrections).

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1 Introduction

Derivation of non-perturbative 'stringy' contributions to effective field theory is one of the major problems in string theory towards its phenomenological applications. In the absence of fundamental formulation of non-perturbative string/M-theory, any explicit example of summing up some non-perturbative corrections may shed light on other cases as well. The non-perturbative low-energy effective action for hypermultiplets in the *Calabi-Yau* (CY) compactified type-IIA superstings is a good place for such calculations, because the relevant moduli space metric in the effective N=2 supergravity is affected by quantum corrections due to D-instantons and five-brane instantons. The former originate from the (Euclidean) D2-branes wrapped about certain (special Lagrangian) 3-cycles in CY threefold, whereas the latter come from the (NS-NS, Euclidean) five-branes wrapped about the entire CY [1].

With this motivation in mind, most attention in the past was devoted to the so-called *universal hypermultiplet* (UH) present in any CY compactification [2, 3, 4, 5]. The UH contains dilaton amongst its bosonic physical components, while it is essentially gravitational in nature (i.e. gravitational corrections cannot be ignored for UH).

The D-instanton quantum corrections to the quantum moduli space metric of a *single matter hypermultiplet* for the CY-compactified type IIA superstrings near a conifold singularity were calculated by Ooguri and Vafa [6]. They found the unique solution consistent with N=2 *rigid* supersymmetry and toric isometry. The solution [6] was interpreted as the infinite D-instanton sum coming from multiple wrappings of the Euclidean D2-branes around the vanishing cycle [6]. The Ooguri-Vafa solution is given by the hyper-Kähler metric in the limit of flat four-dimensional spacetime, i.e. when N=2 supergravity decouples. The hyper-Kähler solution [6] was lifted to a quaternionic solution in curved spacetime of N=2 supergravity in ref. [7].

With almost all known cases being limited to a single hypermultiplet, it is quite natural to turn attention to many hypermultiplets. A generic CY compactification is well known to yield $n = \dim H^{2,1} + 1$ hypermultiplets, where $\dim H^{2,1}$ is a Hodge number of CY. Unfortunately, the geometrical constraints imposed on hypermultiplets by N=2 supersymmetry are weaker for several hypermultiplets when compared to only one. This may, however, be compensated by requiring enough unbroken symmetries. Our calculations in this Letter are performed in the hyper-Kähler limit when both N=2 supergravity and UH are switched off. The five-brane instanton corrections are weighted by powers of e^{-1/g^2}, whereas that of D-instantons are weighted by powers of $e^{-1/g}$, where g is string couping [8]. Hence, when the string coupling

is sufficiently large (as is the case in our study), the five-brane instantons can be suppressed against the D-instantons. As a final simplification, we assume the existence of n tri-holomorphic (i.e. commuting with N=2 supersymmetry) isometries of the D-instanton corrected quantum moduli space metric of n (identical) matter hypermultiplets.

Our paper is organized as follows. In sect. 2 we birefly review the Ooguri-Vafa solution [6] for a single matter hypermutliplet. In sect. 3 we introduce some mathematical tools needed to set up the framework to our calculations. In sect. 4 we give the results of our calculations for the metric. Sect. 5 is our conclusion.

2 Ooguri-Vafa solution

The *Ooguri-Vafa* (OV) solution [6] describes the D-instanton corrected moduli space metric of a single matter hypermultiplet in type-IIA superstrings compactified on a Calabi-Yau threefold of Hodge number $\dim H^{2,1} = 1$, when *both* N=2 supergravity *and* UH are switched off, while five-brane instantons are suppressed. The matter hypermultiplet low-energy effective action in that limit is given by the four-dimensional N=2 supersymmetric non-linear sigma-model that has the four-dimensional OV metric in its target space.

Rigid (or global) N=2 supersymmetry of the non-linear sigma-model requires a hyper-Kähler metric in its target space [9, 10]. The OV metric has a (toric) $U(1)\times U(1)$ isometry by construction [6]. There always exist a linear combination of two commuting abelian isometries that is tri-holomorphic, i.e. it commutes with N=2 rigid supersymmetry [11].

Given any four-dimensional hyper-Kähler metric with a tri-holomorphic isometry ∂_t, it can always be written down in the standard (Gibbons-Hawking) form [12],

$$ds^2_{\rm GH} = \frac{1}{V}(dt+\hat{\Theta})^2 + V(dx^2+dy^2+dz^2) \ , \tag{1}$$

that is governed by *linear* equations,

$$\Delta V = \vec{\nabla}^2 V \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V = 0 \ , \quad \text{almost everywhere}, \tag{2}$$

and

$$\vec{\nabla} V + \vec{\nabla}\times\vec{\Theta} = 0 \ . \tag{3}$$

The one-form $\hat{\Theta} = \Theta_1 dx + \Theta_2 dy + \Theta_3 dz$ is fixed by the 'monopole equation' (3) in terms of the real scalar potential $V(x,y,z)$. Equation (2) means that the function V

is harmonic (away from possible isolated singularities) in three Euclidean dimensions $\mathbf{R}^3$. The singularities are associated with the positions of D-instantons.

Given extra $U(1)$ isometry, after being rewritten in the cylindrical coordinates $(\rho = \sqrt{x^2+y^2},\ \theta = \arctan(y/x),\ \eta = z)$, the hyper-Kähler potential $V(\rho,\theta,\eta)$ becomes independent upon θ. Equation (1) was used by Ooguri and Vafa [6] in their analysis of the matter hypermultiplet moduli space near a conifold singularity. The conifold singularity arises in the limit of the vanishing CY period,

$$\int_{\mathcal{C}} \Omega \to 0 \quad , \tag{4}$$

where the CY holomorphic (nowhere vanishing) 3-form Ω is integrated over a non-trivial 3-cycle $\mathcal{C}$ of CY. The powerful singularity theory [13] can then be applied to study the universal behaviour of the hypermultiplet moduli space near the conifold limit, by resolving the singularity.

In the context of the CY compactification of type IIA superstrings, the coordinate ρ represents the 'size' of the CY cycle $\mathcal{C}$ or, equivalently, the action of the D-instanton originating from the Euclidean D2-brane wrapped about the cycle $\mathcal{C}$. The physical interpretation of the η coordinate is just the expectation value of another (RR-type) hypermultiplet scalar. The cycle $\mathcal{C}$ can be replaced by a sphere S^3 for our purposes, since the D2-branes only probe the overall size of $\mathcal{C}$.

The OV potential V is *periodic* in the RR-coordinate η since the D-brane charges are quantized [1]. We normalize the corresponding period to be 1, as in ref. [6]. The Euclidean D2-branes wrapped m times around the sphere S^3 couple to the RR expectation value on S^3 and thus should produce additive contributions to V, with the factor of $\exp(2\pi i m\eta)$ each.

In the *classical* hyper-Kähler limit, when both N=2 supergravity and all the D-instanton contributions are suppressed, the potential $V(\rho,\eta)$ of a single matter hypermultiplet cannot depend upon η since there is no perturbative superstring state with a non-vanishing RR charge. Accordingly, the classical pre-potential $V(\rho)$ can only be the Green function of the two-dimensional Laplace operator, i.e.

$$V_{\text{classical}} = -\frac{1}{2\pi}\log\rho + \text{const.} \ , \tag{5}$$

whose normalization is also in agreement with ref. [6].

The calculation of ref. [6] to determine the exact D-instanton contributions to the hyper-Kähler potential V is based on the idea [1] that the D-instantons should resolve the singularity of the classical hypermultiplet moduli space metric at $\rho = 0$. A similar

situation arises in the standard (Seiberg-Witten) theory of a quantized N=2 vector multiplet (see e.g., ref. [10] for a review).

Equation (2) formally defines the electrostatic potential V of electric charges of unit charge in the Euclidean upper half-plane (ρ,η), $\rho>0$, which are distributed along the axis $\rho=0$ in each point $\eta=n\in\mathbf{Z}$, while there are no two charges at the same point [6]. A solution to eq. (2) obeying all these conditions is unique,

$$V_{\rm OV}(\rho,\eta)=\frac{1}{4\pi}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{\sqrt{\rho^2+(\eta-n)^2}}-\frac{1}{|n|}\right)+{\rm const.} \tag{6}$$

After Poisson resummation eq. (6) takes the desired form of singularity resolution [6]:

$$V_{\rm OV}(\rho,\eta)=\frac{1}{4\pi}\log\left(\frac{\mu^2}{\rho^2}\right)+\sum_{m\neq 0}\frac{1}{2\pi}e^{2\pi i m\eta}\,K_0\left(2\pi\,|m|\,\rho\right)\ , \tag{7}$$

where the modified Bessel function K_0 of the 3rd kind has been introduced,

$$K_s(z)=\tfrac{1}{2}\int_0^{+\infty}\frac{dt}{t^{s-1}}\exp\left[-\frac{z}{2}\left(t+\frac{1}{t}\right)\right]\ , \tag{8}$$

valid for all ${\rm Re}\,z>0$ and ${\rm Re}\,s>0$, while μ is a constant (modulus).

Inserting the standard asymptotical expansion of the Bessel function K_0 near $\rho=\infty$ into eq. (7) yields [6]

$$\begin{aligned}V_{\rm OV}(\rho,\eta)\ &=\ \frac{1}{4\pi}\log\left(\frac{\mu^2}{\rho^2}\right)+\sum_{m=1}^{\infty}\exp\left(-2\pi m\rho\right)\cos(2\pi m\eta)\times\\ &\quad\times\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}n!\Gamma(-n+\frac{1}{2})}\left(\frac{1}{4\pi m\rho}\right)^{n+\frac{1}{2}}\quad .\end{aligned} \tag{9}$$

The string coupling constant g can be easily reintroduced into eq. (9) by a substitution $\rho\to\rho/g$. The factors of $\exp\left(-2\pi m\rho/g\right)$ in eq. (9) are the contributions due to the multiple D-instantons [6].

The OV potential (6) is given by a (regularized) T-sum over the T-duality transformations, $\eta\to\eta+1$, being applied to the fundamental solution $V_0\equiv\frac{1}{4\pi r}\equiv\frac{1}{4\pi\sqrt{\rho^2+\eta^2}}$ of eq. (2),

$$V_{\rm OV}(\rho,\eta)=A+\sum_{\rm T}V_0(\rho,\eta)=A+\sum_{\rm T}\frac{1}{4\pi\sqrt{\rho^2+\eta^2}}\quad , \tag{10}$$

where A is a constant. The fundamental solution $V_0(\rho,\eta)$ is just the Green function of the three-dimensional Laplace operator Δ in eq. (2).

3 Pedersen-Poon Ansatz

There exists a natural generalization of the Gibbons-Hawking Ansatz (1) to higher-dimensional toric hyper-Kähler spaces, known as the *Pedersen-Poon* (PP) metric [14]. Namely, given n commuting tri-holomorphic isometries of a (real) $4n$-dimensional hyper-Kähler space, there exists a coordinate system (x^i_a, t_i), with $i, j = 1, 2, \ldots, n$ and $a, b, c = 1, 2, 3$, where the hyper-Kähler metric takes the form [14]:

$$ds^2 = U_{ij} dx^i \cdot dx^j + U^{ij}(dt_i + A_i)(dt_j + A_j) \quad . \tag{11}$$

Here the dot means summation over the a-type indices, all metric components are supposed to be independent upon all t_i (thus reflecting the existence of n isometries), $U^{ij} = (U_{ij})^{-1}$, while $U_i = (U_{i1}, \ldots, U_{in})$ and A_i are to be solutions to the generalized monopole (or BPS) equations:

$$\mathcal{R}_{x^i_a x^j_b} = \varepsilon_{abc} \nabla_{x^i_c} U_j \quad \text{and} \quad \nabla_{x^i_a} U_j = \nabla_{x^j_a} U_i \quad , \tag{12}$$

where the field strength $\mathcal{R}$ of the gauge fields A has been introduced. The PP metric (11) can be completely specified by its real PP-potential $F(x, w, \bar{w})$ that generically depends upon $3n$ variables,

$$x^j = x^j_3 \ , \qquad w^j = \frac{x^j_1 + i x^j_2}{2} \ , \qquad \bar{w}^j = \frac{x^j_1 - i x^j_2}{2} \ , \tag{13}$$

because the BPS equations (12) allow a solution [14]

$$U_{ij} = F_{x^i x^j} \quad \text{and} \quad A_j = i \left(F_{w^k x^j} dw^k - F_{\bar{w}^k x^j} d\bar{w}^k \right) \ , \tag{14}$$

provided F itself obeys (almost everywhere) a linear Laplace-like equation [14]

$$F_{x^i x^j} + F_{w^i \bar{w}^j} = 0 \ . \tag{15}$$

The subscripts of F denote partial differentiation with respect to the given variables.

Equation (15) is apparently the multi-dimensional hyper-Kähler generalization of eq. (2). The existence of the PP-potential F essentially amounts to integrability (or linearization) of the BPS equations (12), because the master equation (15) is linear, whose solutions are not difficult to find. For instance, a general solution to eq. (15) can be written down as the contour (C) integral of an arbitrary potential $G(\eta^j(\xi), \xi)$ [15],

$$F = \mathrm{Re} \oint_C \frac{d\xi}{2\pi i \xi} G(\eta^j(\xi), \xi) \ , \quad \text{where} \quad \eta^j(\xi) = \bar{w}^j + x^j \xi - w^j \xi^2 \quad . \tag{16}$$

4 D-instanton sums

Our technical assumptions are essentially the same as in ref. [6], namely,

- periodicity in all x^i with period 1, due to the D-brane charge quantization,
- the classical potential F near a CY conifold singularity should have a logarithmic behaviour (elliptic fibration), being independent upon all x^i (when all $w^i \to \infty$),
- the classical singularity of the metric should be removable, i.e. an exact metric should be complete (or non-singular),
- the metric should be symmetric under the permutation group of n sets of hypermultipet coordinates $(x^j, w^j, \bar{w}^j)$, where $j = 1, 2, \ldots, n$.

The last assumption means that we only consider *identical* matter hypermultiplets. Together with our main assumption about n tri-holomorphic isometries (see e.g., our Abstract and sect. 3), it is going to lead us to an explicit solution.

We first confine ourselves to the case of *two* identical matter hypermultiplets ($n = 2$), and then quote our result for an arbitrary $n > 2$. The problem amounts to finding a solution to the master equations (15) subject to the conditions above. The known OV solution $V_{\rm OV}(|w|, x) \equiv V_{\rm OV}(x, w)$, defined by eqs. (6) or (7), can be used to introduce another function $F(x, w)$ as a solution to two equations

$$F_{xx} = -F_{w\bar{w}} = V_{\rm OV}(x, w) \ . \tag{17}$$

We do not need an explicit solution to eq. (17) in what wollows. Since eqs. (15) are linear, the superposition principle applies to their solutions. It is now straightforward to verify that there exist a 'trivial' solution to eqs. (15), given by a PP potential

$$F_0 = c_0 \left[F(x_1, w_1) + F(x_2, w_2) \right] \quad , \tag{18}$$

where c_0 is a real constant. A particular non-trivial solution in the $n = 2$ case is given by

$$F_{\rm mixed} = c_+ F(x_1 + x_2, w_1 + w_2) + c_- F(x_1 - x_2, w_1 - w_2) \quad , \tag{19}$$

where $c_\pm$ are two other real constants. Though the linear partial differential equation (15) has many other solutions, we argue at the end of this section that the most general solution, satisfying all our requirements, is actually given by

$$F = F_0 + F_{\rm mixed} \quad . \tag{20}$$

To this end, we continue with the particular solution (20).

To write down our result for the hyper-Kähler moduli space metric of two matter hypermultiplets, we merely need the second derivatives of the PP potential, because of eq. (14). Using eqs. (6), (14), (17), (18), (19) and (20), we find

$$
\begin{aligned}
4\pi U_{11} \;=\;\; & c_{+}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{\sqrt{(x_1+x_2-n)^2+(w_1+w_2)(\bar{w}_1+\bar{w}_2)/\lambda^2}}-\frac{1}{n}\right) \\
& + c_{-}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{\sqrt{(x_1-x_2-n)^2+(w_1-w_2)(\bar{w}_1-\bar{w}_2)/\lambda^2}}-\frac{1}{n}\right) \\
& + c_{0}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{\sqrt{(x_1-n)^2+w_1\bar{w}_1/\lambda^2}}-\frac{1}{n}\right)\quad ,
\end{aligned}
\tag{21}
$$

with a modulus parameter λ. The component U_{22} has the the form as that of eq. (21), but the indices 1 and 2 have to be exchanged. The remaining two components of the matrix U in eq. (11) are given by

$$
\begin{aligned}
4\pi U_{12} = 4\pi U_{21} = \;& c_{+}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{\sqrt{(x_1+x_2-n)^2+(w_1+w_2)(\bar{w}_1+\bar{w}_2)/\lambda^2}}-\frac{1}{n}\right) \\
& + c_{-}\sum_{n=-\infty}^{+\infty}\left(\frac{1}{\sqrt{(x_1-x_2-n)^2+(w_1-w_2)(\bar{w}_1-\bar{w}_2)/\lambda^2}}-\frac{1}{n}\right)\quad .
\end{aligned}
\tag{22}
$$

The A-field components in the PP metric (11) are given by the second equation (14).

The physical interpretation of our solution as the infinite D-instanton sum is evident after rewriting it as an asymptotical expansion like that of eq. (9), by using eqs. (7) and (8), now being applied to the two-hypermultiplet metric components (21) and (22). In particular, their classical behavior is given by

$$
U_{11} \;\sim\; \frac{c_{+}}{4\pi}\ln\frac{\lambda^2}{\left|w_1+w_2\right|^2}+\frac{c_{-}}{4\pi}\ln\frac{\lambda^2}{\left|w_1-w_2\right|^2}+\frac{c_{0}}{4\pi}\ln\frac{\lambda^2}{\left|w_1\right|^2}\quad , \tag{23}
$$

and similarly for U_{22} after exchanging the indices 1 and 2, and U_{12} after dropping the last term in eq. (23).

As regards the case of an arbitrary $n>2$, the easiest picture arises in terms of the PP potential F — see sect. 3. We define this potential as a linear combination of $F(x_1+x_2+\ldots+x_n, w_1+w_2+\ldots+w_n)$, $F(-x_1+x_2+\ldots+x_n, -w_1+w_2+\ldots+w_n)$, $\ldots$, $F(-x_1-x_2+\ldots-x_n, -w_1-w_2-\ldots-w_n)$, $F(x_1,w_1)$, $F(x_2,w_2)$, $\ldots$ and $F(x_n,w_n)$, each one being construcuted out of the OV potential, as in eq. (17). Next, we define the metric components of the matrix U_{ij} in eq. (11) by the second derivatives of the total PP potential F, as in eq. (14). The rest is fully straightforward, *cf.* eqs. (21) and (22).

Finally, some comments about the uniqueness of our solution (20) are in order. First, we notice that all functions U^{ij} also obey eq. (15) by our construction (14). Given any other non-trivial solution, it should lead (after Poisson resummation) to *the same* classical behaviour (23), while it should also reduce to the OV solution (sect. 2) for a single hypermultiplet. According to the Poisson resummation formula,

$$\sum_{n=0}^{\infty} f(n) = \sum_{n=0}^{\infty} \tilde{f}(n) \ , \quad \text{where} \quad \tilde{f}(u) = \int_{-\infty}^{+\infty} dv f(v) e^{2\pi i u v} \ , \tag{24}$$

the logarithmic terms can only come from the $n = 0$ term in the sum, e.g. when using eq. (6), choosing $n = v$ in the Fourier integral (24), and then taking $u = 0$ after the Fourier transform.

Hence, a difference between ours and any other solution to the potentials F or U^{ij} cannot contribute to eq. (23). Since eq. (23) is x-independent, we first investigated the class of x-independent real potentials F, without any reference to the metric (11). By solving eq. (15) in this case we found an unique solution subject to our symmetries, namely, the one given by eq. (23). Hence, the anticipated uniqueness of our solution boils down to the uniqueness of the periodic (of period 1) extension of a given w-dependent solution to a solution depending upon both w and x, and having a generic form $\sum_n G(x+n, w)$ with some basic function G. It seems to be quite plausible that the square root in eq. (21) is the *only* basic function to yield the logarithmic terms out of the $n = 0$ term in the D-istanton sum.

5 Conclusion

In our final results (21) and (22) the quantum corrections to the classical (logarithmic) terms (23) are exponentially suppressed by the D-instanton factors (in the semiclassical description). Unlike the case of a single matter hypermultiplet (sect. 2), the multi-hypermultiplet moduli space metric is also sensitive to the phases of w_i (i.e. not only to their absolute values). There are no perturbative (type IIA superstring) corrections, while all D-instanton numbers contribute to the solution.

Our results can also be applied to an explicit construction of *quaternionic* metrics in real $4(n-1)$ dimensions out of known hyper-Kähler metrics in real $4n$ dimensions, which is of particular importance to a description of D-instantons in N=2 supergravity [16]. For instance, the four-dimensional quaternionic manifolds with toric isometry can be fully classified [17] in terms of the PP-potential F obeying eq. (15).

It would be also interesting to connect our results to perturbative superstring calculations in the D-instanton background, which is still a largely unsolved problem.

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